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Entropic Repulsion for Two Dimensional Multi-Layered Harmonic Crystals

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We consider two dimensional lattice free fields (harmonic crystals) and study the asymptotic behavior of the fields under the constraint that each field lies above a hard-wall and is forced to be piled on top of another. This problem is the so-called entropic repulsion and our result extends that of ref. 2 which studied the higher dimensional case.

KEY WORDS: Entropic repulsion; Gaussian field; Gibbs measure; multi-interface phenomena.

1. INTRODUCTION AND RESULT

Under the situation that two distinct pure phases like crystal/vapor coexist in space, hypersurfaces called interfaces are formed and separate these distinct phases at macroscopic level. One of the problems related to such phase separating interface is the study of the effect of a hard-wall. Especially, to analyze the phenomena of the entropic repulsion and also the wetting transition, asymptotic behavior of the interface under the constraint that the interface fluctuates above a hard-wall has been studied for several interface models.

On the other hand, instead of one random interface, some models of two or more random interfaces interacting through the constraint that one interface lies above the other one has been studied (cf. refs. 5 and 6 and references therein). Such model represents the coexistence of three or more phases and arises when we analyze three phases in thermal equilibrium, A, B, and C, and layer of the phase C is developed at the boundary between the phase A and B, in order to lower the surface tension. Then, there are

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two interfaces, one between A and C, and the other one between C and B. For example, the model of the membrane and so on (cf. refs. 13 and 15, etc.). However, as regards the precise asymptotic behavior of each interface, the previous results in the model of two or more random interfaces are not necessarily satisfactory and in this paper, we will adapt Gaussian lattice free fields (harmonic crystals) as the model of random interfaces and investigate this problem.

Let $d \geqslant 2$, $K \in \mathbb{N}$, $V = [-1, 1]^d$ and $V_N = NV \cap \mathbb{Z}^d$. $\phi^i = \{\phi^i_x\}_{x \in V_N} \in \mathbb{R}^{V_N}$, $1 \leqslant i \leqslant K$ denote independent centered Gaussian fields on $\mathbb{R}^{V_N}$ with the same covariance matrix $G_N \equiv (-\Delta_N)^{-1}$ where Δ_N is the discrete Laplacian on $\mathbb{Z}^d$ with Dirichlet boundary condition outside V_N. We denote the common law as P_N. Then, the configuration $\phi = (\phi^1, \phi^2, \ldots, \phi^K)$ is interpreted as the effective modelization of K (discretized) random interfaces embedded in the $d+1$-dimensional space and the spin ϕ^i_x denotes the height of the ith interface at site $x \in V_N$. Its law is given by the product measure $\mathbb{P}^K_N \equiv P_N^{\otimes K}$.

Our problem here is to examine the asymptotic behavior of the fields ϕ^i, $1 \leqslant i \leqslant K$ under the measure $\mathbb{P}^K_N(\cdot \mid \Omega^{K,+}_{N,\varepsilon})$ as $N \to \infty$, where

$$\Omega^{K,+}_{N,\varepsilon} = \{\phi;\, 0 \leqslant \phi^1_x \leqslant \phi^2_x \leqslant \cdots \leqslant \phi^K_x \text{ for every } x \in V_{N,\varepsilon}\},$$

and $V_{N,\varepsilon} = \{x \in V_N;\, \mathrm{dist}(x, V_N^c) \geqslant \varepsilon N\}$, $0 < \varepsilon < 1$. If we consider the case of $K = 1$, the problem is entropic repulsion between one lattice free field and a hard-wall and this problem has been studied by a number of authors (cf. refs. 10, 11, and references therein). One of the main result is the following (cf. refs. 4 and 8 for $d \geqslant 3$ and ref. 3 for $d = 2$):

$$\lim_{N \to \infty} \sup_{x \in V_{N,\varepsilon}} P_N\left(\left|\frac{\phi^1_x}{\sqrt{\log_d(N)}} - \sqrt{4g_d}\right| \geqslant \eta \mid \Omega^{1,+}_{N,\varepsilon}\right) = 0, \tag{1.1}$$

for every $0 < \varepsilon < 1$ and $\eta > 0$, where

$$\log_d(N) = \begin{cases} (\log N)^2 & \text{if } d = 2, \\ \log N & \text{if } d \geqslant 3, \end{cases}$$

and

$$g_d = \begin{cases} \frac{2}{\pi} & \text{if } d = 2, \\ (-\Delta)^{-1}(0, 0) & \text{if } d \geqslant 3. \end{cases}$$

Δ is the discrete Laplacian on $\mathbb{Z}^d$. Namely, the field is pushed to infinity by a hard-wall and its level is $\sqrt{4g_d \log_d(N)}$. Also, entropic repulsion for one lattice field above an i.i.d. random wall instead of a hard-wall was studied by ref. 1.

For the multi-interface case, only the following higher dimensional case result has been obtained by ref. 2. Let $d \geqslant 3$ and $K=2$. Then, it holds that

$$\lim_{N\to\infty} \mathbb{P}_N^2(S_{N,\varepsilon}(\phi^i) \leqslant (\sqrt{4g_d}\,\lambda_i - \eta)\sqrt{\log N} \mid \Omega_{N,\varepsilon}^{2,+}) = 0,$$

and

$$\lim_{N\to\infty} \mathbb{P}_N^2(S_{N,\varepsilon}(\phi^i) \geqslant (\sqrt{4g_d}\,\lambda_i + \eta)\sqrt{\log N} \mid \Omega_{N,\varepsilon}^{2,+}) = 0,$$

for every $0<\varepsilon<1$, $\eta>0$, and $i=1,2$, where $\lambda_1=1$, $\lambda_2=\sqrt{2}+1$, and $S_{N,\varepsilon}(\phi^i)=\frac{1}{|V_{N,\varepsilon}|}\sum_{x\in V_{N,\varepsilon}}\phi_x^i$ denotes the sample mean of the field ϕ^i over $V_{N,\varepsilon}$.

Remark 1.1. Actually, in ref. 2, entropic repulsion for two interfaces with the different covariances has been studied; see also Remark 1.4 below.

Now, we are in the position to state our result for the case of $d=2$. The first result is on the asymptotics of the probability of the event $\Omega_{N,\varepsilon}^{K,+}$.

Theorem 1.1. Let $d=2$. For every $K\in\mathbb{N}$ and $0<\varepsilon<1$, we have

$$\lim_{N\to\infty} \frac{1}{(\log N)^2} \log \mathbb{P}_N^K(\Omega_{N,\varepsilon}^{K,+}) = -2\left(\sum_{i=1}^{K}\lambda_i^2\right) g_2 \mathscr{C}_\varepsilon,$$

where $\lambda_i=\sqrt{2}\,(i-1)+1$, $1\leqslant i\leqslant K$, $\mathscr{C}_\varepsilon=\mathrm{Cap}(V_\varepsilon)=\inf\{\frac{1}{2d}\|\partial f\|_2^2;\ f\in C_0^\infty(V),\ f\geqslant 0,\ f(r)=1 \text{ if } r\in V_\varepsilon\}$ and $V_\varepsilon=\{r\in V;\ \mathrm{dist}(r,V^c)\geqslant\varepsilon\}$.

The corresponding result for the case of $d\geqslant 3$ and $K=2$ was also shown by ref. 2. Using this probability estimate, we obtain the asymptotics of the sample mean of each field under the conditional measure $\mathbb{P}_N(\cdot \mid \Omega_{N,\varepsilon}^{K,+})$ as $N\to\infty$.

Theorem 1.2. Let $d=2$. For every $K\in\mathbb{N}$, $0<\varepsilon<1, \eta>0$, and $1\leqslant i\leqslant K$, we have

$$\lim_{N\to\infty} \mathbb{P}_N^K(S_{N,\varepsilon}(\phi^i) \leqslant (\sqrt{4g_2}\,\lambda_i - \eta)\log N \mid \Omega_{N,\varepsilon}^{K,+}) = 0, \tag{1.2}$$

and

$$\lim_{N\to\infty} \mathbb{P}_N^K(S_{N,\varepsilon}(\phi^i) \geqslant (\sqrt{4g_2}\,\lambda_i+\eta)\log N \mid \Omega_{N,\varepsilon}^{K,+})=0, \tag{1.3}$$

where $\lambda_i=\sqrt{2}\,(i-1)+1$, $1\leqslant i\leqslant K$.

Combining this two dimensional case result with that of ref. 2 for higher dimensional case, although the order of the repulsion of each field does not change, the size of the repulsion between two layered interfaces becomes bigger than one interface and a hard-wall case for arbitrary dimension. Especially, if we consider the event $\{\phi;\, \phi_x^1\leqslant\phi_x^2$ for every $x\in V_{N,\varepsilon}\}$ instead of $\Omega_{N,\varepsilon}^{2,+}$ then the difference of these two fields $\phi^2-\phi^1$ asymptotically behaves $\sqrt{2}\sqrt{4g_d\log_d(N)}$ by the result of refs. 3 and 4 since the field $\phi^2-\phi^1$ is a centered Gaussian with covariance $2G_N$ under $\mathbb{P}_N^2$ and the condition $\{\phi;\, \phi_x^1\leqslant\phi_x^2$ for every $x\in V_{N,\varepsilon}\}$ is just a hard-wall condition for $\phi^2-\phi^1$. Therefore, these results imply that each field does not give up easily its freedom of the fluctuation and two fields are shifted above to keep enough width of the fluctuation by the hard-wall and non-intersecting condition.

The proof of the result will be given in Sections 2 and 3. By considering each difference of layered two interfaces, we can reduce our problem to one interface case and adapt the strategy of ref. 3 which studied the entropic repulsion for one lattice free field with a hard-wall in two dimension. The proof of the lower bound of the asymptotics of the probability is given by a well-known entropy argument. The proof of the probability upper bound and the height lower bound is given by a conditioning argument and we shall use the result of ref. 3 obtained by a multi-scale analysis.

Finally, we give several remarks about the result.

Remark 1.2. In the case of one interface, the pointwise estimate of repulsion (1.1) is obtained by iterating FKG argument thorough the sample mean estimate of repulsion as Theorem 1.2 (cf. ref. 7, Section 3 and ref. 4, Section 4). However, in two or more interfaces case, since the fields $\{\phi_x^1\}_{x\in V_N}$ and $\{\phi_x^2-\phi_x^1\}_{x\in V_N}$ are negatively correlated, we have not obtained the comparison estimate as

$$\mathbb{P}_N^2(\phi_x^i\geqslant a \mid \Omega_N^{2,+}(A))\leqslant \mathbb{P}_N^2(\phi_x^i\geqslant a \mid \Omega_N^{2,+}(B)),$$

for every $A, B\subset V_N$ with $A\subset B$, $x\in A$, $a\geqslant 0$ and $i=1,2$ where

$$\Omega_N^{2,+}(D)=\{\phi;\, 0\leqslant\phi_x^1\leqslant\phi_x^2 \text{ for every } x\in D\},$$

for $D \subset V_N$. This lack of comparison estimate causes us to obtain only the sample mean estimates of repulsion; see also Section 1.3 of ref. 2.

Remark 1.3. For the case of $d \geqslant 3$, only the two interfaces model has been studied in ref. 2. By considering each difference of layered two interfaces, their results for the probability upper bound and the height lower bound can be easily extended for arbitrary finite number of interfaces. However, the lack of the comparison estimate under the conditioned measure as stated in Remark 1.2 affects the proof of the probability lower bound for higher dimensional case. Therefore, only the two interfaces model has been considered in ref. 2. On the other hand, different from the higher dimensional case, a simple entropy argument gives the appropriate lower bound of the probability for two dimensional case (cf. refs. 3 and 4) and the lack of the comparison estimate does not affect for multi-interface model. Therefore, we can treat arbitrary number of interfaces for two dimensional case.

Remark 1.4. It might be natural to consider the different covariance for each interface as the result of ref. 2 for $d \geqslant 3$. For the case that each covariance is proportional to $(-\Delta_N)$, namely the case that ϕ^i, $1 \leqslant i \leqslant K$ are independent centered Gaussian fields with the different covariance $\chi_i(-\Delta_N)$, $\chi_i > 0$, we can easily extend our result even if $d = 2$. However, since our proof relies on that of ref. 3 for the two dimensional one interface case, our result for $d = 2$ is restricted to this nearest neighbor harmonic crystals model.

2. PROOF OF THEOREM 1.1

In the following two sections, we always assume that $d = 2$. The following estimates of the variance will be used throughout this paper (cf. ref. 12). Recall that P_N is a law of the two dimensional centered Gaussian field on $\mathbb{R}^{V_N}$ with covariance matrix G_N.

Lemma 2.1. For every $0 < \varepsilon < 1$, there exists a constant $C_1 > 0$ such that

$$\sup_{x \in V_{N,\varepsilon}} |\mathrm{Var}_{P_N}(\phi_x) - g_2 \log N| \leqslant C_1,$$

and there exists a constant $C_2 > 0$ such that

$$\sup_{x \in V_N} \mathrm{Var}_{P_N}(\phi_x) \leqslant g_2 \log N + C_2.$$

2.1. Proof of Theorem 1.1: Lower Bound

For $\alpha > 0$ and $f \in C_0^\infty(V)$ with $f \geqslant 0$ and $f(r) = 1$ if $r \in V_\varepsilon$, let $\alpha_N = \alpha \log N$ and $\psi^N = \{\psi_x^N\}_{x \in V_N}$, $\psi_x^N = \alpha_N f(\frac{x}{N})$. We define a shifted measure $\tilde{\mathbb{P}}_N^K(d\phi) = \bigotimes_{i=1}^K P_N \circ T_{\lambda_i \psi^N}^{-1}(d\phi^i)$ where T_ψ is a shift operator defined by $T_\psi \phi = \phi + \psi$ and $\lambda_i = \sqrt{2}\,(i-1)+1$, $1 \leqslant i \leqslant K$. Namely, $\tilde{\mathbb{P}}_N^K$ represents a product measure of K independent Gaussian fields $\{\phi_x^i\}_{x \in V_N}$, $1 \leqslant i \leqslant K$ which have the same covariance G_N and the different mean $\lambda_i \psi^N$, respectively. Then, we have

$$H(\tilde{\mathbb{P}}_N^K \mid \mathbb{P}_N^K) = E^{\tilde{\mathbb{P}}_N^K}\left[\log \frac{d\tilde{\mathbb{P}}_N^K}{d\mathbb{P}_N^K}\right] = \sum_{i=1}^K H(P_N \circ T_{\lambda_i \psi^N}^{-1} \mid P_N),$$

where $H(P \mid Q) = E^P[\log \frac{dP}{dQ}]$ denotes a relative entropy of P with respect to Q for two probability measures P and Q.

Now, we have $\Omega_{N,\varepsilon}^{K,+} = \bigcap_{i=1}^K \{\phi; \phi_x^i - \phi_x^{i-1} \geqslant 0$ for every $x \in V_{N,\varepsilon}\}$ and $\phi^i - \phi^{i-1}$ is a Gaussian field on $\mathbb{R}^{V_N}$ with mean $\tilde{\lambda}_i \psi^N$ and covariance $(\tilde{\lambda}_i)^2 G_N$ under $\tilde{\mathbb{P}}_N^K$, where $\tilde{\lambda}_i \equiv \lambda_i - \lambda_{i-1} = 1$ if $i = 1$ and $= \sqrt{2}$ if $i \geqslant 2$. $\phi^0 \equiv 0$ represents a hard-wall. By using these facts and Lemma 2.1, we obtain

$$\begin{aligned}\tilde{\mathbb{P}}_N^K((\Omega_{N,\varepsilon}^{K,+})^c) &\leqslant \sum_{i=1}^K \sum_{x \in V_{N,\varepsilon}} \mathbb{P}_N^K(\phi_x^i - \phi_x^{i-1} < -\tilde{\lambda}_i \alpha \log N) \\ &\leqslant CN^2 \exp\left\{-\frac{\alpha^2 (\log N)^2}{2g_d \log N}\right\},\end{aligned}$$

for some constant $C > 0$ and the right hand side is $o(1)$ if $\alpha > \sqrt{4g_d}$. The positive constant C in the estimates may change from place to place in this paper. Also, by ref. 3, we know that

$$\lim_{N \to \infty} \frac{1}{(\log N)^2} H(P_N \circ T_{\lambda_i \psi^N}^{-1} \mid P_N) = \frac{1}{8} \lambda_i^2 \alpha^2 \, \|\nabla f\|_2^2.$$

Finally, combining these facts with the entropy inequality:

$$\log \frac{P(A)}{Q(A)} \geqslant -\frac{1}{Q(A)} (H(Q \mid P) + e^{-1}),$$

and optimizing the choice of f and α, we obtain the lower bound. ∎

2.2. Proof of Theorem 1.1: Upper Bound

Let $0<\varepsilon<1$. For the proof of the upper bound, we consider the mesoscopic scale of order N^{γ}, $0<\gamma<1$ and the partition of $V_{N,\varepsilon}=\{x\in V_N;\ \mathrm{dist}(x,V_N^c)\geqslant\varepsilon N\}$ into the box with side-length $2N^{\gamma}+1$. For simplicity, we always assume that $N^{\gamma},\varepsilon N$ are integers and $2N^{\gamma}$ divides $N-\varepsilon N+1$. For $j=(j_1,j_2)$, $1\leqslant j_1,j_2\leqslant\frac{N-\varepsilon N+1}{N^{\gamma}}$, a divided mesoscopic scale box is given by

$$\begin{aligned}B_j^{\gamma}=&[-N+\varepsilon N-1+2N^{\gamma}(j_1-1),-N+\varepsilon N-1+2N^{\gamma}j_1]\\&\times[-N+\varepsilon N-1+2N^{\gamma}(j_2-1),-N+\varepsilon N-1+2N^{\gamma}j_2].\end{aligned}$$

We call each box B_j^{γ} just a γ-box and denote by Π_{γ} the set of γ-boxes in $V_{N,\varepsilon}$. Note that each box has a center and the boundaries of neighboring boxes intersect. The set of the whole boundary is given by

$$\bigcup_j\partial B_j^{\gamma}=\left\{-N+\varepsilon N-1+2N^{\gamma}k;\ 0\leqslant k\leqslant\frac{N-\varepsilon N+1}{N^{\gamma}}\right\}^2.$$

$\mathscr{F}_{\gamma}$ denotes the σ-field generated by $\{\phi_x^i;\ x\in\bigcup_j\partial B_j^{\gamma},\ 1\leqslant i\leqslant K\}$. For each γ-box B, x_B represents the center of the box and we will denote $\phi_B^i=E[\phi_{x_B}^i|\mathscr{F}_{\gamma}]$ the conditional expectation of $\phi_{x_B}^i$ with respect to $\mathscr{F}_{\gamma}$. Note that $\phi_{x_B}^i-\phi_B^i$ is a centered Gaussian random variable with variance $G_{N^{\gamma}-1}(0,0)$ under $\mathbb{P}_N^K(\cdot\mid\mathscr{F}_{\gamma})$.

Now, set $M_0=2(\sum_{i=1}^K\lambda_i^2)\,g_d\mathscr{C}_{\varepsilon}$. For every $\eta>0$, $0<\gamma<1$, $L>0$, and $1\leqslant i\leqslant K$, we define an $\mathscr{F}_{\gamma}$-measurable event

$$A_{L,\eta,\gamma}^i=\{\phi;\ |\{B\in\Pi_{\gamma};\ \phi_B^i\leqslant(\sqrt{4g_d}\,\lambda_i-\eta i)\log N\}|\leqslant Li\},$$

where $|\cdot|$ denotes the cardinality of the set. Then, for the proof of the upper bound, it is enough to show that the following two lemmas hold.

Lemma 2.2. For every $0<\varepsilon<1$ and $\eta>0$, there exist $0<\gamma<1$ and $L>0$ such that

$$\mathbb{P}_N^K\left(\left(\bigcap_{i=1}^K A_{L,\eta,\gamma}^i\right)^c\cap\Omega_{N,\varepsilon}^{K,+}\right)\leqslant K\exp\{-(M_0+1)(\log N)^2\}.$$

Lemma 2.3. For every $0<\varepsilon<1$, $\eta>0$, $0<\gamma<1$ and $L>0$, we have that

$$\limsup_{N\to\infty}\frac{1}{(\log N)^2}\log\mathbb{P}_N^K\left(\left(\bigcap_{i=1}^K A_{L,\eta,\gamma}^i\right)\cap\Omega_{N,\varepsilon}^{K,+}\right)\leqslant-M_0.$$

The next lemma can be proved by following the argument of the proof of Lemma 9 of ref. 3 and this plays an important role in the proof of Lemma 2.2 and (1.2).

Lemma 2.4. Let $\phi = \{\phi_x\}_{x \in V_N}$ be a two dimensional centered Gaussian field on $\mathbb{R}^{V_N}$ with covariance matrix given by χG_N, $\chi > 0$. Define an $\mathscr{F}_\gamma$-measurable event

$$A_{L,\eta,\gamma} = \{\phi \in \mathbb{R}^{V_N}; |\{B \in \Pi_\gamma; \phi_B \leqslant (\sqrt{4\chi g_d} - \eta) \log N\}| \leqslant L\}.$$

Then, for every $0 < \varepsilon < 1$, $\eta > 0$ and $M \geqslant 0$, there exists $0 < \gamma_0 < 1$ such that for every $\gamma_0 < \gamma < 1$, there exists $L_0 > 0$ and it holds that

$$P_N((A_{L,\eta,\gamma})^c \cap \{\phi; \phi_x \geqslant 0 \text{ for every } x \in V_{N,\varepsilon}\}) \leqslant 2 \exp\{-M(\log N)^2\},$$

for every $L \geqslant L_0$.

Proof of Lemma 2.2. At first, note that

$$\left(\bigcap_{i=1}^{K} A^i_{L,\eta,\gamma}\right)^c \subset \bigcup_{i=1}^{K} (A^{i-1}_{L,\eta,\gamma} \cap (A^i_{L,\eta,\gamma})^c),$$

where $A^0_{L,\eta,\gamma}$ denotes $(\mathbb{R}^{V_N})^K$. Therefore,

$$\mathbb{P}^K_N\left(\left(\bigcap_{i=1}^{K} A^i_{L,\eta,\gamma}\right)^c \cap \Omega^{K,+}_{N,\varepsilon}\right)$$
$$\leqslant \sum_{i=1}^{K} \mathbb{P}^K_N(A^{i-1}_{L,\eta,\gamma} \cap (A^i_{L,\eta,\gamma})^c \cap \{\phi; \phi^i_x - \phi^{i-1}_x \geqslant 0 \text{ for every } x \in V_{N,\varepsilon}\}).$$

By the definition of $A^i_{L,\eta,\gamma}$, we observe that

$$A^{i-1}_{L,\eta,\gamma} \cap (A^i_{L,\eta,\gamma})^c \subset \{|\{B \in \Pi_\gamma; \phi^i_B - \phi^{i-1}_B \leqslant (\sqrt{4g_d}\,\tilde{\lambda}_i - \eta) \log N\}| \geqslant L\}.$$

$\phi^i - \phi^{i-1}$ is a centered Gaussian field with covariance $(\tilde{\lambda}_i)^2 G_N$ under $\mathbb{P}^K_N$ and $\phi^i_B - \phi^{i-1}_B = E^{\mathbb{P}^K_N}[\phi^i_{x_B} - \phi^{i-1}_{x_B} \mid \mathscr{F}_\gamma]$. Hence we can use Lemma 2.4 and obtain that there exist $0 < \gamma < 1$ and $L > 0$ such that

$$\mathbb{P}^K_N(A^{i-1}_{L,\eta,\gamma} \cap (A^i_{L,\eta,\gamma})^c \cap \{\phi; \phi^i_x - \phi^{i-1}_x \geqslant 0 \text{ for every } x \in V_{N,\varepsilon}\})$$
$$\leqslant \exp\{-(M_0 + 1)(\log N)^2\},$$

for every $1 \leqslant i \leqslant K$. ∎

Proof of Lemma 2.3. Since,

$$\left(\bigcap_{i=1}^{K} A^i_{L,\eta,\gamma}\right) \cap \Omega^{K,+}_{N,\varepsilon} \subset \bigcap_{i=1}^{K} (A^i_{L,\eta,\gamma} \cap \{\phi;\, \phi^i_x \geqslant 0 \text{ for every } x \in V_{N,\varepsilon}\}),$$

we have that

$$\mathbb{P}^K_N\left(\bigcap_{i=1}^{K} A^i_{L,\eta,\gamma} \cap \Omega^{K,+}_{N,\varepsilon}\right)$$
$$\leqslant \prod_{i=1}^{K} P_N(A^i_{L,\eta,\gamma} \cap \{\phi;\, \phi^i_x \geqslant 0 \text{ for every } x \in V_{N,\varepsilon}\}).$$

Then, completely the same way to the proof of Lemma 10 of ref. 3, we can prove the lemma. ▮

3. PROOF OF THEOREM 1.2

3.1. Proof of (1.2)

We will use the same notation to the proof of the upper bound of Theorem 1.1. Also, we need the following notation: for each $z \in \tilde{V}_{N^\gamma} \equiv [-N^\gamma, N^\gamma - 1]^2 \cap \mathbb{Z}^2$, define $V^\gamma_{N,\varepsilon}(z) = (2N^\gamma \mathbb{Z}^2 + z_0 + z) \cap V_{N,\varepsilon}$ where $z_0 = (N^\gamma, N^\gamma) \in \mathbb{Z}^2$. $V^\gamma_{N,\varepsilon}(z)$ are disjoint for each $z \in \tilde{V}_{N^\gamma}$ and we have $V_{N,\varepsilon} = \bigcup_{z \in \tilde{V}_{N^\gamma}} V^\gamma_{N,\varepsilon}(z)$.

Lemma 3.1. For every $0 < \varepsilon < 1, \eta > 0$, and $\delta > 0$, there exists $0 < \gamma < 1$ such that

$$\mathbb{P}^K_N(|\{B \in \Pi_\gamma;\, \phi^i_{x_B} - \phi^{i-1}_{x_B} \leqslant (\sqrt{4g_d}\, \tilde{\lambda}_i - \eta) \log N\}| \geqslant \delta |\Pi_\gamma| \mid \Omega^{K,+}_{N,\varepsilon})$$
$$= o(N^{-2\gamma}),$$

as $N \to \infty$ for every $1 \leqslant i \leqslant K$.

We shall prove this lemma later. Once we have Lemma 3.1, by shifting the partition and the corresponding set of centers, we can obtain the similar estimates. Then, since

$$\{|\{x \in V_{N,\varepsilon};\, \phi^i_x - \phi^{i-1}_x \leqslant (\sqrt{4g_d}\, \tilde{\lambda}_i - \eta) \log N\}| \geqslant \delta\, |V_{N,\varepsilon}|\}$$
$$\subset \bigcup_{z \in \tilde{V}_{N^\gamma}} \{|\{x \in V^\gamma_{N,\varepsilon}(z);\, \phi^i_x - \phi^{i-1}_x \leqslant (\sqrt{4g_d}\, \tilde{\lambda}_i - \eta) \log N\}| \geqslant \delta\, |V^\gamma_{N,\varepsilon}(z)|\},$$

we can prove that

$$\lim_{N\to\infty} \mathbb{P}_N^K(|\{x \in V_{N,\varepsilon};\, \phi_x^i - \phi_x^{i-1} \leqslant (\sqrt{4g_d}\,\tilde{\lambda}_i - \eta)\log N\}| \geqslant \delta\,|V_{N,\varepsilon}|\,|\,\Omega_{N,\varepsilon}^{K,+}) = 0, \tag{3.1}$$

for every $0<\varepsilon<1, \eta>0, \delta>0$ and $1 \leqslant i \leqslant K$. Note that $\{x_B;\, B \in \Pi_\gamma\} = V_{N,\varepsilon}^\gamma(0)$.

On $\Omega_{N,\varepsilon}^{K,+}$, if $\phi_x^i \leqslant \sqrt{4g_d}\,\lambda_i - \eta$, then at least either $\phi_x^{i-1} \leqslant \sqrt{4g_d}\,\lambda_{i-1} - \frac{1}{2}\eta$ or $\phi_x^i - \phi_x^{i-1} \leqslant \sqrt{4g_d}\,\tilde{\lambda}_i - \frac{1}{2}\eta$ holds. Thus, we have

$$\begin{aligned}
&\{|\{x \in V_{N,\varepsilon};\, \phi_x^i \leqslant (\sqrt{4g_d}\,\lambda_i - \eta)\log N\}| \geqslant \delta\,|V_{N,\varepsilon}|\}\\
&\quad\subset \{|\{x \in V_{N,\varepsilon};\, \phi_x^{i-1} \leqslant (\sqrt{4g_d}\,\lambda_{i-1} - \tfrac{1}{2}\eta)\log N\}| \geqslant \tfrac{1}{2}\delta\,|V_{N,\varepsilon}|\}\\
&\quad\cup \{|\{x \in V_{N,\varepsilon};\, \phi_x^i - \phi_x^{i-1} \leqslant (\sqrt{4g_d}\,\tilde{\lambda}_i - \tfrac{1}{2}\eta)\log N\}| \geqslant \tfrac{1}{2}\delta\,|V_{N,\varepsilon}|\}.
\end{aligned}$$

Now, by using induction with respect to i and (3.1), we can prove the following lemma and this yields (1.2) (cf. proof of (2.4) of ref. 14).

Lemma 3.2. For every $0<\varepsilon<1, \eta>0, \delta>0$ and $1 \leqslant i \leqslant K$, we have that

$$\lim_{N\to\infty} \mathbb{P}_N^K(|\{x \in V_{N,\varepsilon};\, \phi_x^i \leqslant (\sqrt{4g_d}\,\lambda_i - \eta)\log N\}| \geqslant \delta\,|V_{N,\varepsilon}|\,|\,\Omega_{N,\varepsilon}^{K,+}) = 0.$$

Proof of Lemma 3.1. Let $0<\varepsilon<1, \eta>0, \delta>0, L>0, 0<\gamma<1$, and $1 \leqslant i \leqslant K$. Define

$$\begin{aligned}
J_{\eta,\gamma}^i &= \{B \in \Pi_\gamma;\, \phi_{x_B}^i - \phi_{x_B}^{i-1} \leqslant (\sqrt{4g_d}\,\tilde{\lambda}_i - \eta)\log N\},\\
\tilde{J}_{\eta,\gamma}^i &= \{B \in \Pi_\gamma;\, \phi_B^i - \phi_B^{i-1} \leqslant (\sqrt{4g_d}\,\tilde{\lambda}_i - \eta)\log N\}.
\end{aligned}$$

Recall that $\phi_B^i = E^{\mathbb{P}_N^K}[\phi_{x_B}^i \mid \mathscr{F}_\gamma]$ for a mesoscopic scale box $B \in \Pi_\gamma$. We also define events $F_{\delta,\eta,\gamma}^i = \{|J_{\eta,\gamma}^i| \geqslant \delta\,|\Pi_\gamma|\}$ and $\tilde{A}_{L,\eta,\gamma}^i = \{|\tilde{J}_{\eta,\gamma}^i| \leqslant L\}$. Then, we have

$$\begin{aligned}
&\mathbb{P}_N^K(|\{B \in \Pi_\gamma;\, \phi_{x_B}^i - \phi_{x_B}^{i-1} \leqslant (\sqrt{4g_d}\,\tilde{\lambda}_i - \eta)\log N\}| \geqslant \delta\,|\Pi_\gamma|\,|\,\Omega_{N,\varepsilon}^{K,+})\\
&\quad= \mathbb{P}_N^K(F_{\delta,\eta,\gamma}^i \cap \tilde{A}_{L,\frac{1}{2}\eta,\gamma}^i \mid \Omega_{N,\varepsilon}^{K,+}) + \mathbb{P}_N^K(F_{\delta,\eta,\gamma}^i \cap (\tilde{A}_{L,\frac{1}{2}\eta,\gamma}^i)^c \mid \Omega_{N,\varepsilon}^{K,+}).
\end{aligned}$$

On $F^i_{\delta,\eta,\gamma} \cap \tilde{A}^i_{L,\frac{1}{2}\eta,\gamma}$, we have that $|J^i_{\eta,\gamma} \cap (\tilde{J}^i_{\frac{1}{2}\eta,\gamma})^c| \geqslant \delta\,|\Pi_\gamma| - L$ and if $B \in J^i_{\eta,\gamma} \cap (\tilde{J}^i_{\frac{1}{2}\eta,\gamma})^c$ then $|(\phi^i_{x_B} - \phi^i_B) - (\phi^{i-1}_{x_B} - \phi^{i-1}_B)| \geqslant \frac{1}{2}\eta \log N$. This yields

$$\begin{aligned}
&\mathbb{P}^K_N(F^i_{\delta,\eta,\gamma} \cap \tilde{A}^i_{L,\frac{1}{2}\eta,\gamma} \mid \Omega^{K,+}_{N,\varepsilon}) \\
&\quad \leqslant \frac{1}{\mathbb{P}^K_N(\Omega^{K,+}_{N,\varepsilon})} \mathbb{P}^K_N\left(\sum_{B \in \Pi_\gamma} I\left(|(\phi^i_{x_B} - \phi^i_B) - (\phi^{i-1}_{x_B} - \phi^{i-1}_B)| \geqslant \frac{1}{2}\eta \log N\right)\right. \\
&\qquad\qquad \left. \geqslant \delta\,|\Pi_\gamma| - L\right).
\end{aligned}$$

Under $\mathbb{P}^K_N(\cdot \mid \mathscr{F}_\gamma)$, $\{(\phi^i_{x_B} - \phi^i_B) - (\phi^{i-1}_{x_B} - \phi^{i-1}_B);\, B \in \Pi_\gamma\}$ are i.i.d. centered Gaussian random variables with variance $(\tilde{\lambda}_i)^2\, G_{N^\gamma - 1}$. Now, set $\theta^i_B \equiv I(|(\phi^i_{x_B} - \phi^i_B) - (\phi^{i-1}_{x_B} - \phi^{i-1}_B)| \geqslant \frac{1}{2}\eta \log N)$. Then $\{\theta^i_B;\, B \in \Pi_\gamma\}$ are i.i.d. random variables under $\mathbb{P}^K_N(\cdot \mid \mathscr{F}_\gamma)$ and by a Gaussian estimate and Lemma 2.1, we see that $E^{\mathbb{P}^K_N}[\theta^i_B \mid \mathscr{F}_\gamma] \leqslant \exp\{-C\eta^2 \log N\}$ for some constant $C > 0$. Therefore, $\sum_{B \in \Pi_\gamma} (\theta^i_B - E^{\mathbb{P}^K_N}[\theta^i_B \mid \mathscr{F}_\gamma]) \geqslant \frac{1}{2}\delta\,|\Pi_\gamma|$ for every N large enough if $\sum_{B \in \Pi_\gamma} \theta^i_B \geqslant \delta\,|\Pi_\gamma| - L$. Combining these facts, we obtain

$$\begin{aligned}
&\mathbb{P}^K_N(F^i_{\delta,\eta,\gamma} \cap \tilde{A}^i_{L,\frac{1}{2}\eta,\gamma} \mid \Omega^{K,+}_{N,\varepsilon}) \\
&\quad \leqslant \frac{1}{\mathbb{P}^K_N(\Omega^{K,+}_{N,\varepsilon})} \mathbb{P}^K_N\left(\mathbb{P}^K_N\left(\sum_{B \in \Pi_\gamma} (\theta^i_B - E^{\mathbb{P}^K_N}[\theta^i_B \mid \mathscr{F}_\gamma]) \geqslant \tfrac{1}{2}\delta\,|\Pi_\gamma| \mid \mathscr{F}_\gamma\right)\right) \\
&\quad \leqslant \exp\{M_0(\log N)^2\} \exp\{-C\,|\Pi_\gamma|\} \\
&\quad \leqslant \exp\{-CN^{2(1-\gamma)-\kappa}\},
\end{aligned}$$

for some $\kappa > 0$ and $C > 0$, where we used Lemma 11 of ref. 3 and Theorem 1.1 for the second inequality.

On the other hand, we have

$$\begin{aligned}
&\mathbb{P}^K_N(F^i_{\delta,\eta,\gamma} \cap (\tilde{A}^i_{L,\frac{1}{2}\eta,\gamma})^c \mid \Omega^{K,+}_{N,\varepsilon}) \\
&\quad \leqslant \frac{1}{\mathbb{P}^K_N(\Omega^{K,+}_{N,\varepsilon})} \mathbb{P}^K_N((\tilde{A}^i_{L,\frac{1}{2}\eta,\gamma})^c \cap \{\phi^i_x - \phi^{i-1}_x \geqslant 0 \text{ for every } x \in V_{N,\varepsilon}\}).
\end{aligned}$$

By using Theorem 1.1 and Lemma 2.4, for every $0 < \varepsilon < 1$ and $\eta > 0$, there exist $0 < \gamma < 1$ and $L > 0$ such that

$$\begin{aligned}
&\mathbb{P}^K_N(F^i_{\delta,\eta,\gamma} \cap (\tilde{A}^i_{L,\frac{1}{2}\eta,\gamma})^c \mid \Omega^{K,+}_{N,\varepsilon}) \\
&\quad \leqslant \exp\{M_0(\log N)^2\} \exp\{-(M_0+1)(\log N)^2\} \\
&\quad = \exp\{-(\log N)^2\},
\end{aligned}$$

for every $1 \leqslant i \leqslant K$. Therefore, we can complete the proof. ∎

3.2. Proof of (1.3)

For the proof of (1.3), we shall prove the following lemma.

Lemma 3.3. For every $0<\varepsilon<1$ and $1\leqslant i\leqslant K$, we have that

$$\limsup_{N\to\infty}\frac{1}{\log N}E^{\mathbb{P}_N^K}[S_{N,\varepsilon}(\phi^i)\,|\,\Omega_{N,\varepsilon}^{K,+}]\leqslant\sqrt{4g_d}\,\lambda_i,$$

where $\lambda_i=\sqrt{2}\,(i-1)+1$.

Combining this lemma with Chebyshev inequality and (1.2), we can obtain (1.3).

Proof of Lemma 3.3. We shall use the same notation to the proof of the lower bound of Theorem 1.1 and prove a slightly better statement

$$\limsup_{N\to\infty}\sup_{x\in V_N}\frac{1}{\log N}E^{\mathbb{P}_N^K}[\phi_x^i\,|\,\Omega_{N,\varepsilon}^{K,+}]\leqslant\sqrt{4g_d}\,\lambda_i, \tag{3.2}$$

for every $0<\varepsilon<1, \eta>0$ and $1\leqslant i\leqslant K$. At first, we have a stochastic domination in the FKG sense $\mathbb{P}_N^K(\cdot)\prec\tilde{\mathbb{P}}_N^K(\cdot)$ as the probability measure on $(\mathbb{R}^{V_N})^K$ and by approximating the condition $\Omega_{N,\varepsilon}^{K,+}$ by a convex potential, we also have the stochastic domination $\mathbb{P}_N^+(\cdot\,|\,\Omega_{N,\varepsilon}^{K,+})\prec\tilde{\mathbb{P}}_N^K(\cdot\,|\,\Omega_{N,\varepsilon}^{K,+})$ (cf. ref. 11, Appendix B.1). Then, in the same way to the proof of the upper bound of Theorem 4 of ref. 3, we have

$$\begin{aligned}E^{\mathbb{P}_N^K}[\phi_x^i\,|\,\Omega_{N,\varepsilon}^{K,+}]&\leqslant E^{\tilde{\mathbb{P}}_N^K}[\phi_x^i\,|\,\Omega_{N,\varepsilon}^{K,+}]\\&\leqslant\frac{1}{\tilde{\mathbb{P}}_N^K(\Omega_{N,\varepsilon}^{K,+})}(\lambda_i\alpha\log N+\sqrt{G_N(x,x)}).\end{aligned}$$

Therefore, by the proof of Theorem 1.1 lower bound and Lemma 2.1, we obtain the lemma. ∎

Remark 3.1. In the case of $d=2$, by using (3.2) and Brascamp–Lieb inequality which can be applied to the conditioned measure $\mathbb{P}_N^K(\cdot\,|\,\Omega_{N,\varepsilon}^{K,+})$ (cf. Appendix of ref. 9), the argument of the proof of the upper bound of Theorem 4 of ref. 3 yields a slightly better result

$$\lim_{N\to\infty}\sup_{x\in V_N}\mathbb{P}_N^K(\phi_x^i\geqslant(\sqrt{4g_d}\,\lambda_i+\eta)\log N\,|\,\Omega_{N,\varepsilon}^{K,+})=0,$$

for every $0<\varepsilon<1, \eta>0$ and $1\leqslant i\leqslant K$.

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